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Policies for inventory/distribution systems: The effect of centralization vs. decentralization

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Abstract

This paper concerns with a multi-echelon inventory/distribution system considering one-warehouse and N-retailers. The retailers are replenished from the warehouse. We assume that the demand rate at each retailer is known. The problem consists of determining the optimal reorder policy which minimizes the overall cost, that is, the sum of the holding and replenishment costs. Shortages are not allowed and lead times are negligible. We study two situations: when the retailers make decisions independently and when the retailers are branches of the same firm. Solution methods to determine near-optimal policies in both cases are provided. Computational results on several randomly generated problems are reported.

Keywords: Inventory control; Inventory/distribution systems; Multi-echelon systems

1. Introduction

The multi-echelon inventory/distribution systems represent a special category of inventories encountered in practice where several installations are involved. Due to their applicability in real-world situations, the multi-echelon systems have caught the researchers' attention. A special type of such inventory systems deals with one-warehouse and N-retailers (e.g., a chain of stores supplied by a single regional warehouse). In this problem, the warehouse is the sole supplier of N retailers. Customer demand occurs at each retailer at a constant rate. This demand must be met as it occurs over an infinite horizon without either shortages or backlogging. Orders placed by retailers generate demands at the warehouse. There is a holding cost rate per unit stored per unit time and a fixed charge for each order placed at the warehouse and at each retailer. The demand rates, holding unit costs, and setup costs are stationary and facility dependent. Delivery of orders is assumed to be instantaneous, that is, lead times are assumed to be zero. The goal is to find a policy with minimum or near-minimum long-run average cost.

This one-warehouse and N-retailers system was examined by Schwarz (1973) and he showed that the form of the optimal policy can be very complex; in particular, it requires that the order quantity at one or more of the locations varies with time even though all relevant demand and

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cost factors are time invariant. Thus, he considered the possibility of restricting the class of strategies, where the order quantity at each location does not change with time and he determined the necessary conditions of an optimal policy. Moreover, he provided a solution method for the one-warehouse and N identical retailers problem and suggested heuristic solutions for the general case. Schwarz (1973) proved that an optimal policy can be found in the set of "basic" policies. A basic policy is any feasible policy where deliveries are made to the warehouse only when the warehouse has zero inventory and, at least one retailer has zero inventory. Moreover, deliveries are made to any given retailer only when that retailer has zero inventory. In addition, all deliveries made to any given retailer between successive deliveries to the warehouse are of equal size.

In particular, Schwarz (1973) introduced two classes of basic policies: myopic and single cycle policies and he also tested the near optimality of these policies using three lower bounds. A myopic policy is one which optimizes a given objective function with respect to any two stages and it ignores multi-stage interaction effects. Accordingly, the one-warehouse and N-retailers system is viewed as N one-warehouse and one-retailer systems. A single cycle policy is one that is stationary and nested. A policy is said to be stationary if each facility always orders the same quantity at equally spaced points in time. A nested policy is one in which each time any stage orders, all of its successors also order.

Graves and Schwarz (1977) performed a similar analysis for arborescent systems in which each stage obtains its supply from an unique immediate predecessor and supplies its output to a set of immediate successors. They reduced the class of admissible policies for stationary continuous-time infinite-horizon multi-stage production/inventory problems to find a good approximation to optimal policies, presenting a branch-and-bound algorithm to determine optimal single cycle policies for arborescent systems. They also examined the near-optimality of the myopic policies.

Roundy (1985) showed that nested policies can have very low effectiveness in the worst case and he defined new classes of policies for the one-warehouse N-retailers problem: q-optimal integer-ratio and optimal power-of-two policies. He proved that for any data set, the effectiveness of q-optimal integer-ratio and optimal power-of-two policies is at least 94% and 98%, respectively.

In this paper, we introduce near-optimal policies for inventory/distribution systems with one-warehouse and N-retailers considering an instantaneous demand pattern at the warehouse. We study two cases: if the warehouse and the retailers belong to the same firm (centralization), or if the warehouse and retailers belong to different firms (decentralization).

Outside customer demand rates are assumed known and constant. Shortages and lead times are not allowed. At each stage, a fixed-order cost which is independent of the lot size and a holding unit cost are considered. The goal consists of determining the near-optimal policy with minimum overall cost both when there exists dependence and when not.

We introduce a solution method to obtain the near-optimal plan in the case of independence or decentralization. This method allows us to know in advance the number of periods of the demand vector at the warehouse. Once this number is calculated, either the Wagner and Whitin (1958) algorithm or the Wagelmans et al. (1992) procedure for inventory systems with time-varying demand can be applied. On the other hand, when the N retailers are branches of the same firm (centralization), we deal with the class of single cycle policies and we propose a branch and bound algorithm to determine the near-optimal plan.

The outline of the remaining of this paper is divided into seven sections. In Section 2, we introduce the notation and terminology required to state the problem. Section 3 is devoted to the one-warehouse and N-retailers problem assuming that the retailers are independent of the warehouse. In such a situation, the problem becomes a time-varying demand inventory system and a procedure to determine the number of periods at the warehouse is provided. Section 4 deals with the centralized situation, that is, when the retailers and the warehouse belong to the same firm. In this case, two different policies are suggested: the retailers can place their orders at a common replenishment time or at different replenishment times. In the former case, the solution can be obtained directly using an analytical approach. In the latter case, a solution method based on a branch and bound scheme is proposed. Section 5 presents a numerical example which is analyzed assuming that there is both centralization and decentralization among the warehouse and the retailers. Computational results are reported in Section 6. Finally, in Section 7, we present our conclusions and final remarks.

2. Terminology and problem statement

Consider a multi-echelon inventory/distribution system where a warehouse supplies N retailers. Assume that customer demand occurs at each retailer at a constant rate. This demand must be met as it occurs without shortages. Orders placed by retailers generate demands at the warehouse. There is a holding cost per unit stored per unit time and a fixed charge for each order placed at the warehouse and at each retailer. The demand rates, holding unit costs, and replenishment costs are stationary and facility dependent. Deliveries of orders are assumed to be instantaneous.

Hereafter we use the following notation:

D_j	demand per unit time at retailer j, $j = 1, \ldots, N$
$\overline{D_w}$	demand vector at the warehouse (for decentralized decisions)
D_w	demand per unit time at the warehouse (for centralized decisions)
k_j	fixed ordering cost of a replenishment at retailer j, $j = 1, \ldots, N$
k_w	fixed ordering cost of a replenishment at the warehouse
h_j	unit carrying cost at retailer j, $j = 1, \ldots, N$
h_w	unit carrying cost at the warehouse
t_j	time interval between replenishments at retailer j, $j = 1, \ldots, N$
$\overline{t_w}$	vector that contains the time instants where the retailers place their orders to the warehouse (for decentralized decisions)
τ_w	time horizon at the warehouse (for decentralized decisions)
t_w	time interval between replenishments at the warehouse (for centralized decisions)
Q_j	order quantity at retailer j, $j = 1, \ldots, N$
$\overline{Q_w}$	order quantities vector at the warehouse (for decentralized decisions)
Q_w	order quantity at the warehouse (for centralized decisions)
C_j	total cost at retailer j, $j = 1, \ldots, N$
C_w	total cost at the warehouse
C	overall cost of the firm

The aim consists of minimizing the overall cost, that is, the sum of holding and replenishment costs at the warehouse and at the retailers. In general, the cost function is

$$C = C_w + \sum_{j=1}^{N} C_j = C_w + \sum_{j=1}^{N} \left(k_j \frac{D_j}{Q_j} + h_j \frac{Q_j}{2} \right). \tag{1}$$

Depending on whether there exists dependence or not among the warehouse and the retailers, the cost function at the warehouse is formulated in a different way.

From this point on, we propose a procedure which determines near-optimal solutions for the decentralized case.

3. Decentralized case

Suppose that there exists independence among the retailers and the warehouse. In such a situation, it is assumed that each installation belongs to different firms. For this reason, each retailer is interested in minimizing its own cost independently.

Let $C_j(Q_j)$ be the cost function at retailer j, $C_j(Q_j) = h_j Q_j/2 + k_j D_j/Q_j$. Since the previous formula stands for the cost function for a EOQ system, we can determine the optimal lot size, Q_j^*, the optimal planning time, t_j^*, and the optimal cost, C_j^*, using the following classical expressions, that is,

$$Q_j^* = \sqrt{\frac{2D_jk_j}{h_j}}, \quad t_j^* = \frac{Q_j^*}{D_j} \text{ and } \quad C_j^* = \sqrt{2D_jk_jh_j}$$

for $j = 1, \ldots, N$.

Since each retailer places orders according to an EOQ pattern, the planning times are not related. Therefore, the warehouse behaves as an inventory system with time-varying demand. When the demand rate varies with time, we can no longer assume that the best strategy is to always order the same replenishment quantity. In fact, this will seldom be the case. Hence, the warehouse does not follow the classical saw-teeth pattern of the EOQ model. Indeed, we now have to use the demand information at the retailers, over a finite planning horizon, to determine the appropriate replenishment quantities at the warehouse.

Following Schwarz (1973), deliveries are made to the warehouse only when the warehouse and at least one retailer have zero inventory. Note that the optimal planning times for each retailer are real values. Therefore, we cannot assure that a point in time exists where all the retailers order simultaneously. In this case, the number of periods of the demand vector at the warehouse is not finite and, the problem cannot be solved by the Wagelmans et al. algorithm (1992). Under this assumption, we propose an approach to overcome this problem. The idea consists of either truncating or rounding up to rational times the real planning times. It is clear that the solution provided by this method is not the real optimal plan but it is quite a good approximation. Furthermore, in practice, it does not make sense to work with irrational times.

Let B be the set of rational times where any retailer orders to the warehouse, that is, $B = \{t \in \mathbb{Q}: t = nt_i, \text{ for some } n \in \mathbb{N} \text{ and } i \in (1, 2, \ldots, N)\}$, where each $t_i = a_i/b_i$, $i = 1, 2, \ldots, N$, is obtained by rounding or truncating the optimal planning time at each retailer. Moreover, following the characterization of the "basic" policies stated by Schwarz (1973), each value in B represents a candidate instant where the warehouse can place an order.

Since the optimal planning times have been transformed into rational values, a set $S = \{n_1, n_2, \ldots, n_N\}$ of integer values can always be found such that $n_1t_1 = n_2t_2 = \cdots = n_Nt_N = \tau_w$, or, in other words

$$n_1\frac{a_1}{b_1} = n_2\frac{a_2}{b_2} = \cdots = n_N\frac{a_N}{b_N} = \tau_w. \tag{2}$$

Recall that τ_w or an integer multiple of τ_w represents the planning time for the warehouse.

Note that (2) represents a linear equations system with n variables and $n-1$ equations. In order to assure the integrality of the n_i's, set

$$n_N = b_Na_{N-1}a_{N-2}\cdots a_2a_1. \tag{3}$$

Therefore, the remaining integer values in (2) are obtained by

$$n_i = n_N\frac{a_N}{b_N}\frac{b_i}{a_i}, \quad i = 1, 2, \ldots, N-1. \tag{4}$$

Finally, each n_i's is divided by the m.c.d.$(n_1, n_2, \ldots, n_N)$. Then, the values thus obtained are considered as the new n_i's values and τ_w can be calculated by (2). Also, these values can be used to determine the number P of different instants in time over τ_w where the warehouse receives an order from some retailer. First of all, the values n_i's must be clustered in the following way. Those n_j's that are powers of some n_i value are included in a cluster. That is, $n_j = n_i^k$, for some k integer. If there are not n_j's values that are powers of some n_i, then this cluster contains only the n_i value. Let R be the number of clusters. For each cluster i, we choose the highest power value n_i' as the representative element. That is, $n_i' = n_i^k$ being k the highest power. Then, set $m_i = n_i' - 1$ for $i = 1, 2, \ldots, R$. The integer m_i represents the number of equidistant points over τ_w needed to get n_i' intervals. The theorem below states when orders are placed to the warehouse. The proof of Theorem 2 requires the following lemma.

Lemma 1. *Let t_1 and t_2 be two rational numbers and let n_1 and n_2 be integer numbers such that $n_1t_1 = n_2t_2 = \tau_w$. Then, the number of points in $(0, \tau_w)$ where $it_1 = jt_2$ for $i = 1, 2, \ldots, n_1$ and $j = 1, 2, \ldots, n_2$ is given by the* m.c.d.$(n_1, n_2) - 1$.

The proof of this lemma is straightforward.

Theorem 2. *The number P of different instants in time where the warehouse receives an order from some retailer is*

$$P = \sum_{i=1}^{R} m_i - \sum_{i=1}^{R-1} \sum_{j=i+1}^{R} (\text{m.c.d.}(n'_i, n'_j) - 1). \qquad (5)$$

Proof. By Lemma 1, the double summation in (5) represents the points in $(0, \tau_w)$ which have been considered more than once in the first summation. Therefore, P stands for the number of different instants in $(0, \tau_w)$ where the warehouse receives an order from some retailer. □

Once the number of points P is obtained, we can generate the demand vector at the warehouse of dimension $P+1$ or a multiple of $P+1$. Since the planning times have been rounded the order quantity at each retailer, Q_j^*, has changed to be $Q_j = t_j D_j$. Let J_j be the set of retailers ordering from the warehouse in period j. This set can be used to determine the quantity to be satisfied by the warehouse in period j, $j = 0, 1, \ldots, P$, in the following way $D_w[j] = \sum_{i \in J_j} Q_i$. This demand vector represents the quantities that the warehouse has to supply. To solve this problem, the Wagelmans et al. algorithm (1992) can be used.

Below, we present a numerical example with five retailers and one warehouse where the planning times at each retailer are

Retailer 1	Retailer 2	Retailer 3	Retailer 4	Retailer 5
$t_1 = 5$	$t_2 = 5/2$	$t_3 = 5/3$	$t_4 = 10/3$	$t_5 = 10/9$

Then, using the method shown above, the values in S are $n_1 = 2, n_2 = 4, n_3 = 6, n_4 = 3, n_5 = 9$ and $\tau_w = 10$.

The next step consists of clustering these values including into a cluster all the n_j's values that are powers of some n_i value in S. Hence, the representative elements of each cluster are: $n'_1 = 4$, $n'_2 = 9$, $n'_3 = 6$ and, therefore, $m_1 = 3$, $m_2 = 8$, $m_3 = 5$. Then, according to Theorem 2, P is equal to 13 and the number of periods at the warehouse is 14.

Now, we have to determine the demand ordered from the warehouse in each interval, that is, $\overline{D_w}$, and then we apply the Wagelmans et al. algorithm (1992). See the numerical example presented in Section 5 for more details.

The following section is devoted to the centralized case. Under this assumption, we study different policies.

4. Centralized case

In this case, the warehouse and the retailers belong to the same firm. Therefore, the firm should pay all the costs, and the goal is to minimize (1), that is, the cost at the warehouse plus the costs at the retailers. Taking into account that the firm has to make decisions about the stock control, it can force the retailers to place their orders at the same time or to place an order either at different instants of time or independently. The latter case was already studied in the previous section.

4.1. Assuming common replenishment time

To coordinate the replenishments, the firm can force the retailers to place their orders at the same time, say every t time units.

Then, the cost at each retailer j, $j = 1, \ldots, N$, is

$$C_j = h_j \frac{D_j t}{2} + \frac{k_j}{t}.$$

Let D be the sum of the demands at the retailers, that is, $D = \sum_{j=1}^{N} D_j$. Since all retailers place an order at the same time, the one-warehouse N-retailers problem can be viewed as a one-warehouse one-retailer problem where the demand per unit time at the warehouse is $D_w = D$. Besides, the new retailer orders the sum of the quantities ordered by the retailers, that is, $Q = \sum_{j=1}^{N} Q_j$ every t time units.

For that reason, the order quantity Q_w at the warehouse can be determined according to the integer-ratio policy. Crowston et al. (1973) and Williams (1982) proved the optimality of the integer-ratio policy for two-echelon systems. Therefore, in our problem, we can follow this integer-ratio policy and set $Q_w = nQ$, where n is a positive integer. Then, the cost at the warehouse, assuming instantaneous demand pattern, is $C_w = h_w((n-1)/2)tD + k_w/nt$.

The aim consists of minimizing the sum of total holding cost plus ordering cost at the warehouse and at the retailers, that is,

$$C(t,n) = \frac{t}{2}\left(\sum_{j=1}^{N} h_j D_j + h_w(n-1)D\right) + \frac{1}{t}\left(\sum_{j=1}^{N} k_j + \frac{k_w}{n}\right).$$

Note that the overall cost depends on t and n only. To calculate the optimal solution (t^*, n^*) we need the following Lemma.

Lemma 3. *If* $h_w < h_j$, $j = 1, \ldots, N$, *and* n *is a continuous variable, then* $C(t,n)$ *is convex over the region*: $\{R\colon 0 < n < \infty, 0 < t \leqslant B(n)\}$, *and has its global minimum at* (t^*, n^*), *where*

$$B(n) = \frac{1}{n}\left[\frac{2k_w}{h_w D}\left[2\left(1 + n\frac{\sum_{j=1}^{N} k_j}{k_0}\right)^{1/2} - 1\right]\right]^{1/2}$$

and

$$t^* = \left[\frac{2\left(\sum_{j=1}^{N} k_j + \frac{k_w}{n}\right)}{\sum_{j=1}^{N} h_j D_j + h_w(n-1)D}\right]^{1/2}, \tag{6}$$

$$n^* = \left[\frac{k_w\left(\sum_{j=1}^{N} h_j D_j - h_w D\right)}{h_w D \sum_{j=1}^{N} k_j}\right]^{1/2}. \tag{7}$$

Proof. Assuming that n is a continuous variable and setting the first partial derivatives of $C(t,n)$ equal to zero, we obtain t^* by (6) and n^* by (7).

It is easy to see that the Hessian is positive definite at $t = t^*$ and $n = n^*$, therefore, $C(t,n)$ has a local minimum at (t^*, n^*).

The Hessian matrix is non-negative definite for any n and t in the region: $\{R\colon 0 < n < \infty, 0 < t \leqslant B(n)\}$. Also, $(t^*, n^*) \in R$. Thus, $C(t,n)$ is convex on R with the global minimum at (t^*, n^*). □

From the value t^*, we can obtain the optimal order quantities for each retailer, that is,

$$Q_j^* = D_j t^*, \quad j = 1, \ldots, N \tag{8}$$

and

$$Q_w^* = n \sum_{j=1}^{N} Q_j^*, \tag{9}$$

where n is the nearest integer to n^*.

Summarizing, if the firm forces the retailers to place their orders at the same time, the optimal solution is given by the formulae in Table 1.

However, due to some reasons such as logistics problems, it could be preferable to satisfy the demand at the retailers at different time instants. Hence, in the following section, we develop the case where the retailers can place orders at different instants of time.

4.2. Assuming different replenishment times

The firm can allow the retailers to place their orders at different instants of time t_j, $j = 1, 2, \ldots, N$. This case concerns the class of single cycle policies, and hence, the unique condition that must be verified is that there must exist $n_1, n_2, \ldots, n_N \in \mathbb{N}$, such that, $n_1 t_1 = n_2 t_2 = \cdots = n_N t_N = t_w$.

Schwarz (1973) was the first who considered this class of policy. He provided an optimal solution for the one-warehouse and N identical retailers problem (transforming the system into a one-warehouse one-retailer system), and suggested a heuristic solution for the general one-warehouse N-retailers problem. However, this heuristic does not always provide good solutions. Graves and Schwarz (1977) proposed a solution method to get optimal single cycle policies for this problem. Roundy (1985) focused on the special class of single cycle policies known as power-of-two policies. He proved that the effectiveness of power-of-two policies with fixed base planning period, is at least 94%. That is, when we restrict ourselves to such policies, we can guarantee a solution whose cost is at most 6% above the cost of an optimal policy. Furthermore, if the base planning period is assumed to be variable,

Table 1

	Time	Quantity
Retailer 1	$t_1 = t^* = \left[\dfrac{2(\sum_{j=1}^{N} k_j + k_w/n)}{(\sum_{j=1}^{N} h_j D_j + h_w(n-1)D)}\right]^{1/2}$	$Q_1^* = D_1 t^*$
Retailer 2	$t_2 = t^*$	$Q_2^* = D_2 t^*$
$\vdots$	$\vdots$	$\vdots$
Retailer N	$t_N = t^*$	$Q_N^* = D_N t^*$
Warehouse	$t_w = nt^*$	$Q_w^* = n\sum_{j=1}^{N} Q_j^*$

the power-of-two policies are at least 98% effective.

We propose a procedure which combines the relaxation of the integrality constraint of the n_i's, along with a branch and bound scheme. This approach runs with lower computational effort than Graves and Schwarz's method (1977). Moreover, as it is shown in Section 6, this procedure generates better single cycle policies than those provided by Roundy's method (1985) when stationary and nested policies are considered.

The new approach assumes that, in period t_w, retailer j has to order n_j times and it holds $n_j D_j t_j^2/2 = t_w t_j D_j/2$ units of item. Therefore, the cost at retailer j is as follows:

$$C_j = 1/t_w(h_j(t_w t_j D_j/2) + k_j n_j) = h_j t_j D_j/2 + k_j/t_j.$$

On the other hand, the warehouse only places an order once, and it holds $t_w \sum_{j=1}^{N} D_j(t_w - t_j)/2$ units of item. Thus, the cost at the warehouse is given by

$$C_w = \frac{1}{t_w}\left(h_w t_w \sum_{j=1}^{N} \frac{D_j(t_w - t_j)}{2} + k_w\right) = h_w \sum_{j=1}^{N} \frac{D_j(t_w - t_j)}{2} + \frac{k_w}{t_w}.$$

Therefore, to find the optimal single cycle policy we have to solve

$$\min C(t_w, t_1, t_2, \ldots, t_N) = \frac{k_w}{t_w} + h_w \frac{t_w \sum_{j=1}^{N} D_j}{2} + \sum_{j=1}^{N}\left(\frac{k_j}{t_j} + (h_j - h_w)\frac{D_j t_j}{2}\right) \tag{10}$$

$$\text{s.t.} \quad n_i t_i = n_j t_j = t_w, \quad i,j = 1,2,\ldots,N, \tag{11}$$

$$n_j \geqslant 1, \quad \text{integer}, \quad j = 1,2,\ldots,N.$$

The first step to solve (10) consists of relaxing the integrality constraint of the n_j's. Then, the optimal replenishment times that minimize (10) are $t_w = [2k_w/h_w D]^{1/2}$, being $D = \sum_{j=1}^{N} D_j$, and $t_j = [2k_j/(h_j - h_w)D_j]^{1/2}$, $j = 1, \ldots, N$.

Taking into account (11), the optimal n_j's values can be calculated as

$$n_j = \frac{t_w}{t_j} = \left[\frac{k_w(h_j - h_w)D_j}{k_j h_w D}\right]^{1/2}.$$

Unfortunately, in general, these n_j's are not integers. However, we propose a solution method based on a branch and bound scheme to obtain near-optimal integer n_j's from the real values.

We start sorting the retailers so that retailer i is smaller than retailer j, if $n_i < n_j$. We can assume, without loss of generality, that $n_1 < n_2 < \cdots < n_N$.

Then, we proceed to find the near-optimal integers by generating an initial feasible solution setting n_j equals to the nearest integer value, or

$n_j = 1$ if $n_j < 1$, $j = 1, 2, \dots, N$. This feasible solution provides an upper bound, *UB*, for the total cost which allows us to ignore worse solutions than the *UB*.

Afterward, the procedure generates an enumeration tree with N levels where each level corresponds to a different n_j, $j = 1, 2, \dots, N$. At level $i+1$, each n_j, $j = 1, 2, \dots, i$, is fixed and only n_k's, $k = i+1, \dots, N$, have to be determined.

Note that if n_j's, $j = 1, \dots, i$, are known, each reorder time t_j has changed to satisfy $t_j = n_j t_{\mathrm{w}}$.

Thus, the total cost can be reformulated as follows:

$$\begin{aligned}
C(t_{\mathrm{w}}, t_{i+1}, \dots, t_N) &= \frac{k_{\mathrm{w}}}{t_{\mathrm{w}}} + h_{\mathrm{w}} \frac{t_{\mathrm{w}} D}{2} \\
&\quad + \sum_{j=1}^{i} \left(k_j \frac{n_j}{t_{\mathrm{w}}} + (h_j - h_{\mathrm{w}}) \frac{D_j}{2} \frac{t_{\mathrm{w}}}{n_j} \right) \\
&\quad + \sum_{j=i+1}^{N} \left(\frac{k_j}{t_j} + (h_j - h_{\mathrm{w}}) \frac{D_j t_j}{2} \right) \\
&= \frac{1}{t_{\mathrm{w}}} \left(k_{\mathrm{w}} + \sum_{j=1}^{i} k_j n_j \right) \\
&\quad + \frac{t_{\mathrm{w}}}{2} \left(h_{\mathrm{w}} D + \sum_{j=1}^{i} (h_j - h_{\mathrm{w}}) \frac{D_j}{n_j} \right) \\
&\quad + \sum_{j=i+1}^{N} \left(\frac{k_j}{t_j} + (h_j - h_{\mathrm{w}}) \frac{D_j t_j}{2} \right).
\end{aligned}$$

Let t_{w}^i denote the optimal reorder time at the warehouse assuming that n_j's, $j = 1, \dots, i$, are known, that is,

$$t_{\mathrm{w}}^i = \left[\frac{2(k_{\mathrm{w}} + \sum_{j=1}^{i} k_j n_j)}{h_{\mathrm{w}} D + \sum_{j=1}^{i} (h_j - h_{\mathrm{w}}) D_j / n_j} \right]^{1/2}.$$

Once we know t_{w}^i and taking into account (11), we can calculate the new optimal n_{i+1} as

$$n_{i+1} = \frac{t_{\mathrm{w}}^i}{t_{i+1}}. \tag{12}$$

Thus, considering that n_j's, $j = 1, 2, \dots, i$, are known, the minimum cost given by the above procedure is $C(t_{\mathrm{w}}, t_{i+1}, \dots, t_N)$. This cost represents a lower bound LB for the subproblem where the n_j's, $j = 1, 2, \dots, i$, are known and integer-valued. If this lower bound exceeds the upper bound UB, the subproblem does not need to be examined. In the opposite case, if $C(t_{\mathrm{w}}, t_{i+1}, \dots, t_N)$ does not exceed the upper bound UB, then, the subproblem is branched at level $i+1$ generating two new subproblems. The first corresponds to set $n_{i+1} = \lfloor n_{i+1} \rfloor$ and the second corresponds to set $n_{i+1} = \lfloor n_{i+1} \rfloor + 1$, where n_{i+1} is the real value determined from (12). For each subproblem the previous procedure is applied.

When the cost associated with a feasible solution at level N is lower than the current upper bound UB, we update UB to be the new cost which has been calculated and the procedure continues looking for a better solution.

Finally, when the branch and bound stops, we can assure that each n_j is an integer value and the replenishment time at the warehouse is given by

$$t_{\mathrm{w}}^N = \left[\frac{2(k_{\mathrm{w}} + \sum_{j=1}^{N} k_j n_j)}{h_{\mathrm{w}} D + \sum_{j=1}^{N} (h_j - h_{\mathrm{w}}) D_j / n_j} \right]^{1/2}.$$

Once we know $n_1, n_2, \dots, n_N$ and t_{w}^N, the replenishment time at each retailer is computed using (11). Given the t_j's, we can calculate the quantity ordered by retailer j as $Q_j = D_j t_j$. Finally, it is easy to see that the order quantity at the warehouse is $Q_{\mathrm{w}} = \sum_{j=1}^{N} n_j Q_j$.

The computational experience shows that the procedure is quite fast since the lower bound for each subproblem allows us to ignore a lot of possible branches in the enumeration tree.

The following section illustrates the different solution methods for both decentralized and centralized cases.

5. Numerical example

Consider a numerical example with three-retailers and one-warehouse with the input data given in Table 2.

Now we proceed to calculate the optimal costs provided by the three policies introduced in the previous sections.

Table 2
Input data

	D_i	k_i	h_i
Retailer 1	75	42	48
Retailer 2	79	100	21
Retailer 3	97	28	52
Warehouse		37	8

Table 3
Optimal order quantities and planning times

	Q_i^*	t_i^*
Retailer 1	$\sqrt{\frac{2\cdot75\cdot42}{48}} \simeq 11.4564$	$\frac{\sqrt{2\cdot75\cdot42/48}}{75} \simeq 0.1527$
Retailer 2	$\sqrt{\frac{2\cdot79\cdot100}{21}} \simeq 27.4295$	$\frac{\sqrt{2\cdot79\cdot100/21}}{79} \simeq 0.3472$
Retailer 3	$\sqrt{\frac{2\cdot97\cdot28}{52}} \simeq 10.2206$	$\frac{\sqrt{2\cdot97\cdot28/52}}{97} \simeq 0.1053$

Table 4
Order quantities and costs at each retailer

	Q_i	C_i
Retailer 1	15.0	570.0000
Retailer 2	23.7	582.1833
Retailer 3	9.7	532.2000

5.1. Assuming decentralization

Using the classical EOQ expressions, we can calculate the optimal order quantities and planning times (Table 3).

As you can see, the planning times are not rational numbers. For that reason, we round the t_i^*'s to obtain the following values: $t_1 = 0.2 = \frac{2}{10}$, $t_2 = 0.3 = \frac{3}{10}$ and $t_3 = 0.1 = \frac{1}{10}$. Now, the n_i's values can be calculated using (3) and (4) to give $n_1 = 60 \cdot \frac{1}{10} \cdot \frac{10}{2} = 30$, $n_2 = 60 \cdot \frac{1}{10} \cdot \frac{10}{3} = 20$ and $n_3 = 10 \cdot 3 \cdot 2 = 60$.

Then, we divide the n_i's values by the m.c.d.$(n_1, n_2, n_3) = 10$ obtaining the following results: $n_1 = 3$, $n_2 = 2$ and $n_3 = 6$. After that, the different clusters are calculated. In this case there are three clusters, one for each n_j. Hence, $n'_j = n_j$, $j = 1, 2, 3$.

Using the new planning times, the order quantities and the costs at each retailer are given in Table 4.

The planning time is $\tau_w = n_i t_i = 0.6$. The number P of instants where the warehouse receives an order is

$$\sum_{i=1}^{3} m_i - \sum_{i=1}^{3-1} \sum_{j=i+1}^{3} (m.c.d.\ (n'_i, n'_j) - 1) = 5$$

and the time vector $\overline{t_w}$ is

0.0	0.1	0.2	0.3	0.4	0.5

The demand vector at the warehouse $\overline{D_w}$ is given by

48.4	9.7	24.7	33.4	24.7	9.7

Once the demand vector is obtained, the Wagelmans et al. algorithm (1992) provides the optimal order planning for the warehouse. That is,

$\overline{Q_w} =$

58.1	0.0	58.1	0.0	34.4	0.0

The cost at the warehouse is 255.4 \$/time unit. The overall cost including the costs at the retailers and at the warehouse is 1939.7833 \$/time unit.

5.2. Centralization with common replenishment time

In this case, the retailers place their orders at the same time. Using (6) and (7), we have $t^* = 0.2004$ time units and $n^* = 0.9482$ and, therefore, $n = 1$. Thus, the retailers and the warehouse place their orders once every $t^* = 0.2004$ time units. The order quantities at the retailers are calculated using (8). Accordingly, $Q_1^* = 15.03$ units of item, $Q_2^* = 15.83$ units of item and $Q_3^* = 19.44$ units of item. Then, the order quantity at the warehouse can be computed from (9) to give $Q_w^* = 50.30$ units of item. Following this policy the overall cost is 2065.2947 \$/time unit.

5.3. Centralization with different replenishment times

Now, the retailers can place their orders at different times t_j, $j = 1, 2, 3$, subject to the

Table 5

	Roundy's procedure			New approach		
	n_i	t_i	Q_i	n_i	t_i	Q_i
Retailer 1	2	0.1441	10.8075	2	0.1608	12.0600
Retailer 2	1	0.2882	22.7678	1	0.3216	25.4064
Retailer 3	2	0.1441	13.9777	3	0.1072	10.3984
Warehouse	1	0.2882	72.3382	1	0.3216	80.7216

constraint $n_1t_1 = n_2t_2 = n_3t_3 = t_w$, where $n_1, n_2, n_3 \in \mathbb{N}$. The new approach introduced in Section 4.2 and Roundy's procedure (1985) can be applied. Below, we show the optimal stationary and nested power-of-two policy given by Roundy's procedure, and the policy provided by the new approach (Table 5).

When Roundy's procedure is used, the overall cost is 1922.1409 \$/time unit. In contrast, when the new approach is applied, the overall cost is 1906.3500 \$/time unit. Therefore, the solution obtained using the new approach is better than Roundy's solution (1985). For this example, this solution is also better than the policies generated by the procedures in Sections 3 and 4.1. Unfortunately, we cannot assure that the centralized policy (with different replenishment times) is always better than the decentralized one. There are instances where the best solution is obtained when the retailers make decisions independently.

The computational experience developed in the next section shows that the new approach always provides policies equal to or better than those given by Roundy's approach. Moreover, we will see that as the number of retailers increases so does the number of instances where the new procedure generates better solutions than Roundy's method.

6. Computational results

Before starting with the comparison analysis between centralized and decentralized policies, we should choose the approach to be implemented in the centralized case. We have carried out a computational experience consisting of 100 instances, where the parameters D_j, k_w, h_w and k_j vary uniformly in the interval $[1, 100]$ and the value h_j is selected from a uniform distribution in $[h_w + 1, 101]$. The results summarized in Table 6 show that the new procedure introduced in Section 4.2 always provides policies equal to or better than those given by Roundy's method (1985). This is due to the fact that solutions provided by Roundy's procedure are confined to power-of-two policies, while the new approach generates integer policies which are not limited by the power-of-two constraint. The first row in this table represents the number of retailers N and the second row contains the number of instances where both methods provide the same solution. Finally, the last row provides the number of problems where the new procedure is better than Roundy's approach. The results in Table 6 suggest that we should use the new procedure instead of Roundy's method.

Once we have determined that the new procedure is better than Roundy's method, we proceed to compare this approach with the decentralized method proposed in Section 3. In this analysis, the number of retailers N takes the following values: $2, 3, 4, 5, 6, 7, 8, 9, 10, 15, 20, 25$ and 30. The parameters D_j, k_w, h_w and k_j have been chosen from two different uniform distributions varying on $[1, 100]$ and on $[1, 10]$, respectively. Moreover, given h_w, the value h_j is selected from a uniform distribution on $[h_w + 1, 101]$ and on $[h_w + 1, 11]$, respectively. For each problem, 100 instances were carried out and the results are shown in Table 7. The first column represents the number of retailers. The results in the second and third columns are obtained when D_j, k_w, h_w and k_j are selected from a $U[1, 100]$ and h_j from a $U[h_w + 1, 101]$. In contrast, the results in the fourth and fifth columns are obtained when D_j, k_w, h_w and k_j are selected from a $U[1, 10]$ and h_j from a $U[h_w + 1, 11]$.

Table 6
Comparison between Roundy's procedure and the new approach when D_j, k_w, h_w and k_j are selected from a uniform distribution on $[1, 100]$ and h_j from a uniform distribution on $[h_w + 1, 101]$

N	2	3	4	5	6	7	8	9	10	15	20	25	30
$C_{NA} = C_R$	85	76	69	66	64	62	54	47	48	46	32	14	10
$C_{NA} < C_R$	15	24	31	34	36	38	46	53	52	54	68	86	90

C_R denotes the cost of the policy computed using Roundy's method and C_{NA} represents the cost of the solution provided by the new approach.

In particular, the second column collects the number of instances where the decentralized approach provides better costs than the centralized case, and the third column shows the number of instances where the centralized case is better. When parameters range in $[1, 100]$, the average number of instances where it is preferable to apply the centralized policy, assuming different replenishment times, is around 45%. On the other hand, when parameters vary on $[1, 10]$, the average number of instances where it is preferable to apply the centralized policy is around 52%. However, these percentages change depending on the number of retailers. For example, for $N = 2$ and considering the first interval, it is better to assume the centralized policy in 87% of instances. Nevertheless, for $N = 20$ and considering the same interval, the best solution is always given by the decentralized approach.

From Table 7, it is easy to see that as the number of retailers increases, so does the number of instances where the decentralized policy is better. However, the gap between this number and the one corresponding to the centralized case decreases when the parameters vary in the interval $[1, 10]$. In our opinion, this fact can be explained since the variability of the parameters is reduced from [1,100] to [1,10]. Conversely, the reduction of the interval leads the demands and costs of the retailers to be quite similar. For that reason, in some instances the centralized policy gives better solutions even when $N = 20$.

In order to analyze the effect of the parameters, a more detailed analysis is required. Accordingly, the number of retailers is fixed to 10 and the parameters are chosen from different uniform distributions, which are shown in the first three columns in Table 8. For each combination, 10 problems are tested. The fourth, sixth and eighth columns in Table 8, contain the number of instances where the decentralized approach provides better policies than the centralized case. In contrast, the fifth, seventh and ninth columns show the number of instances where the centralized case is better.

Table 8 shows that as the interval of the replenishment cost at the warehouse increases so does the number of instances where the centralized policy provides better solutions. On the other hand, when the costs at the retailers are significantly greater than the costs at the warehouse, it is preferable that the retailers make decisions independently.

Table 7
Comparison between decentralized and centralized policies with different replenishment times

N	$D_j, k_w, h_w, k_j \sim U[1, 100]$ $h_j \sim U[h_w + 1, 101]$		$D_j, k_w, h_w, k_j \sim U[1, 10]$ $h_j \sim U[h_w + 1, 11]$	
	Dec.	Cent.	Dec.	Cent.
2	13	87	10	90
3	20	80	17	83
4	43	57	30	70
5	37	63	43	57
6	33	67	37	63
7	50	50	47	53
8	50	50	50	50
9	63	37	37	63
10	43	57	53	47
15	67	33	47	53
20	100	0	70	30
25	100	0	80	20
30	100	0	100	0

Table 8
Comparison between decentralized and centralized policies with different replenishment times when k_w, h_w are selected from the uniform distributions: $U_1 \equiv U[1, 10]$, $U_2 \equiv U[10, 100]$, and $U_3 \equiv U[100, 1000]$

k_w	h_w	h_j	$k_j \sim U_1$		$k_j \sim U_2$		$k_j \sim U_3$	
			Dec.	Cent.	Dec.	Cent.	Dec.	Cent.
U_1	U_1	U_4	5	5	4	6	10	0
U_1	U_2	U_5	5	5	10	0	7	3
U_1	U_3	U_6	7	3	6	4	10	0
U_2	U_1	U_4	0	10	6	4	8	2
U_2	U_2	U_5	0	10	5	5	7	3
U_2	U_3	U_6	0	10	1	9	5	5
U_3	U_1	U_4	0	10	0	10	3	7
U_3	U_2	U_5	0	10	0	10	1	9
U_3	U_3	U_6	0	10	0	10	2	8

h_j's are selected from the uniform distributions: $U_4 \equiv U[h_w + 1, 101]$, $U_5 \equiv U[h_w + 1, 1001]$, and $U_6 \equiv U[h_w + 1, 10001]$.

In Tables 7 and 8, we have only shown the ratio where either the decentralized or centralized policy is better, but nothing is told about the difference between the costs of both procedures. In Table 9, we report a collection of 25 instances, where parameters D_j, k_w, h_w and k_j vary in [1, 100] and h_j in $[h_w + 1, 101]$. The first column represents the number of retailers with $N = 3, 5, 10, 15$ and 20. For the decentralized case, the cost of each instance is shown in the second column. The next two columns contain the costs for the centralized case assuming common and different replenishment times, respectively. For each instance, footnote a indicates the smallest cost. In the last column, the gap (%) represents the quotient between the difference of the costs in the second and fourth column and the minimum of them.

Table 9
Comparison among costs using the different policies for several instances

N	Cost for the decentralized case	Cost for the centralized case (common times)	Cost for the centralized case (different times)	Gap (%)
3	4381.57	4374.70	4252.46[a]	3
3	3719.15	3547.20	3547.20[a]	4
3	1523.50[a]	1600.41	1600.41	5
3	3763.06	3831.73	3663.96[a]	2
3	1573.08	1545.73	1540.28[a]	2
5	6739.30	6926.62	6512.45[a]	3
5	4185.26	4178.21	4009.00[a]	4
5	5277.57[a]	5788.62	5678.70	7
5	3695.20[a]	3881.59	3881.59	2
5	5732.79[a]	5966.72	5918.62	3
10	8064.20	7503.95	7470.76[a]	7
10	8669.57[a]	8996.98	8867.98	2
10	6419.84	6747.98	6176.58[a]	3
10	7564.83	7717.85	7322.83[a]	3
10	7083.12[a]	7247.13	7225.19	2
15	13955.70[a]	15032.60	15015.90	7
15	9433.32[a]	10126.52	9904.33	5
15	13483.20[a]	14935.30	14064.90	8
15	16597.80	17032.40	16415.10[a]	1
15	8337.82	8172.11	8172.11[a]	2
20	14427.30[a]	15883.00	15623.77	8
20	11082.80[a]	12352.40	11932.30	7
20	13419.60[a]	14605.50	14335.00	6
20	9719.26[a]	11224.80	10484.90	7
20	14801.40[a]	17334.30	16893.00	14

[a] Indicates the smallest cost.
The gap (%) represents the quotient between the difference of the costs in the second and fourth column and the minimum of them.

7. Conclusions and final remarks

In this paper, we have studied the one warehouse and N-retailers problem, where stocking decisions have to be adopted to achieve an optimal plan. We have focused our attention on the decentralized and the centralized cases. We have implemented an algorithm to obtain near-optimal ordering plans at the warehouse when the decentralization is addressed. Also, when the centralized case is assumed, we have devised two procedures considering a common replenishment time and different reorder times, respectively.

When the parameters are generated using the same uniform distribution, the results show that as the number of retailers increases so does the number of instances where the decentralized policy is better.

In addition, given a number of retailers, we have carried out an analysis of sensitivity of the parameters. This analysis suggests that, under specific conditions of the unit replenishment and holding costs at the warehouse, the centralized policy can provide better solutions.

Our future research will be focused on the one-warehouse and N-retailers system assuming shortages at the warehouse or at the retailers. Another relevant aspect consists of determining inventory policies for more general structures where several warehouses can deliver goods to different retailers.

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